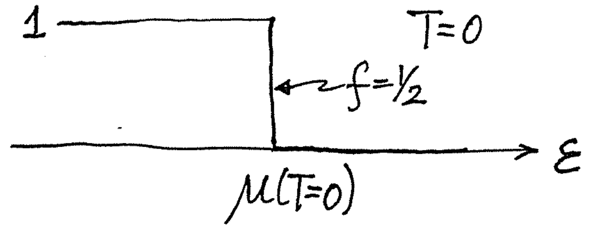
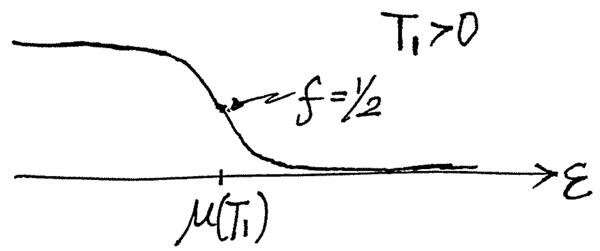
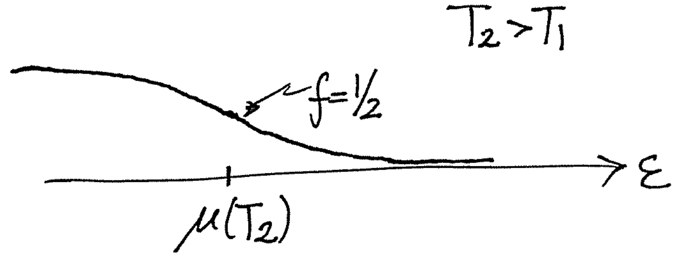
B. Key Features of Fermi-Dirac Distribution

$$f_{FD}(\varepsilon) = \frac{1}{e^{\frac{\varepsilon-\mu}{kT}}+1} = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$

- At the particular energy $\varepsilon=\mu$, $\boxed{f_{FD}(\varepsilon=\mu)=\tfrac{1}{2}}$
- $f_{FD}(\varepsilon)$ behaves "symmetrically" about $\varepsilon=\mu$.







- $f_{FD}(\varepsilon)$ = # fermion in a single-particle state at energy ε (if there is a state at that energy)
- $\mu(T)$ is determined by

$$N = \sum_{\text{s.p. states } i} \frac{1}{e^{\beta(\varepsilon_i-\mu)}+1} = \int_0^\infty g(\varepsilon)\frac{1}{e^{\beta(\varepsilon-\mu)}+1}\,d\varepsilon$$

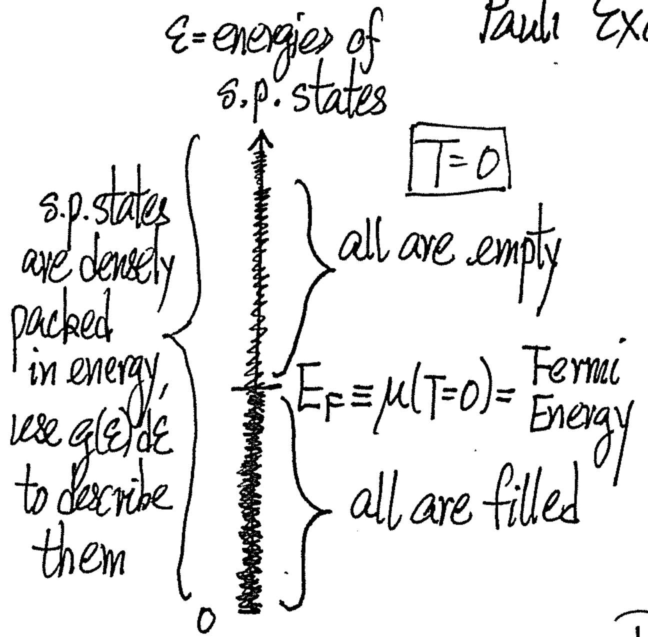
C. T=0 Physics

- T=0 physics dominates Fermi Gas physics ($\because$ Pauli principle)
- This case is referred to as "completely degenerate Fermi Gas"
- Physical Picture

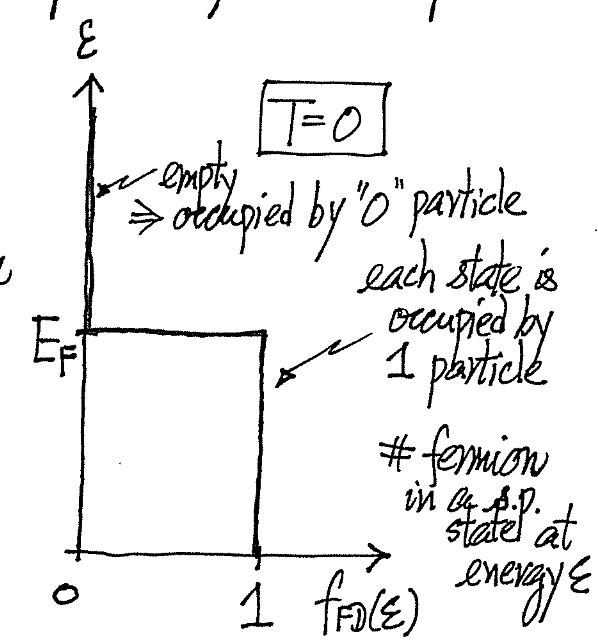
T=0 $\Rightarrow$ Ground state of the whole gas

$\Rightarrow$ fill fermions into s.p. states to get at minimum total energy

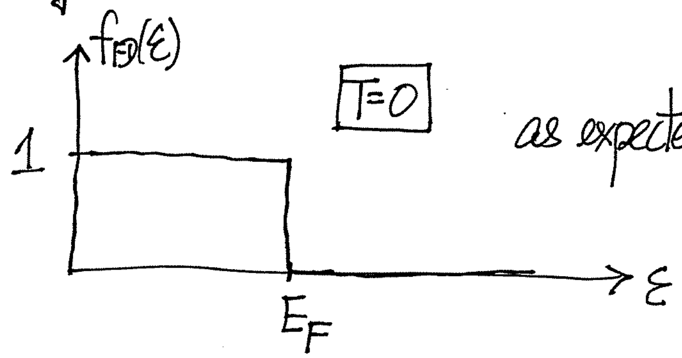
$\Rightarrow$ fill s.p. states one-by-one according to Pauli Exclusion Principle by the N particles



describing this situation by $f_{FD}(\varepsilon)$



Rotate figure



as expected.

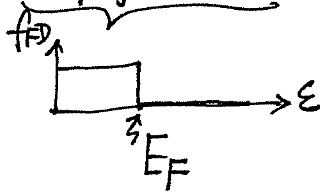
$\boxed{\text{Q: How to determine } \mu(T=0) \text{ or } E_F\text{?}}$

Fermi Energy E_F OR $\mu(T=0)$

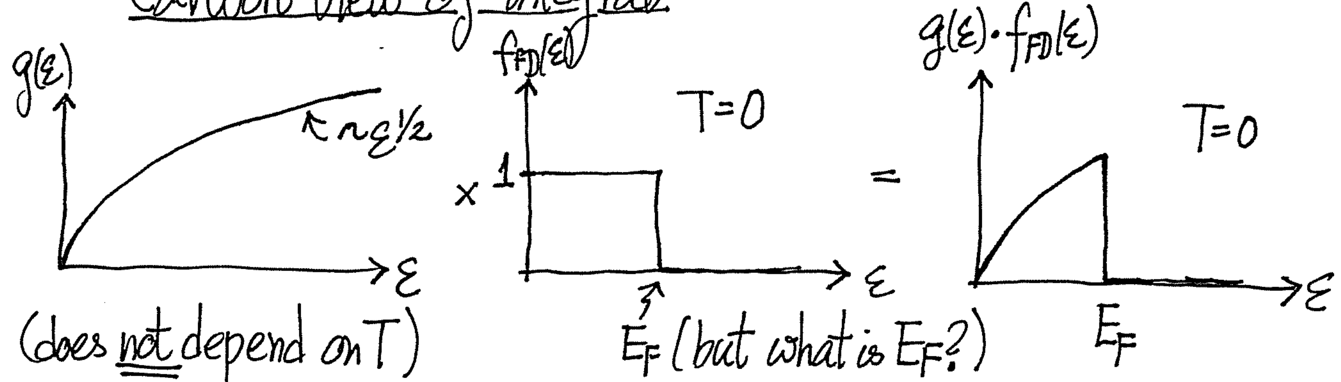
$$N=\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{\varepsilon^{1/2}}{e^{\frac{\varepsilon-\mu}{kT}}+1}\,d\varepsilon \quad \text{fixes } \mu(T) \quad (3)$$

T=0

$$N=\underbrace{\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}}_{g(\varepsilon)}\int_0^\infty \underbrace{\varepsilon^{1/2}}_{g(\varepsilon)}\cdot\underbrace{[\text{Step function}]}_{f_{FD}}\,d\varepsilon \qquad (C1)$$

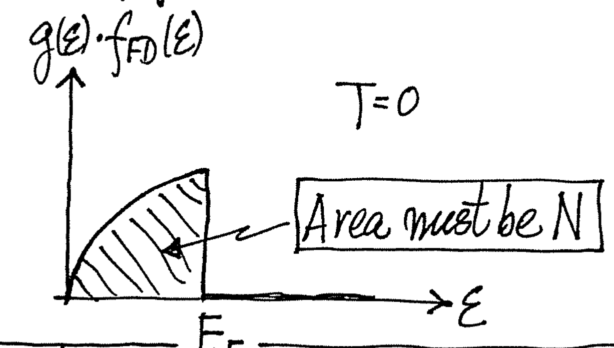


Cartoon view of integral



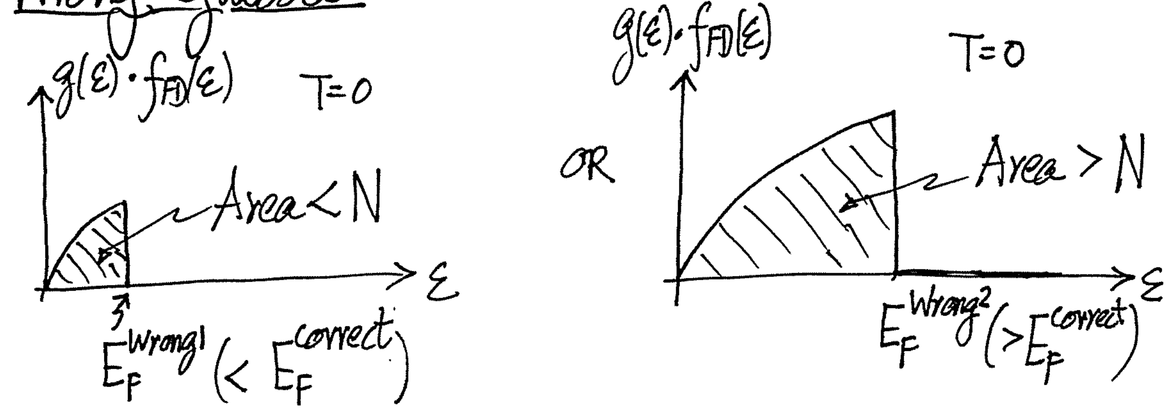
Integration: $\int_0^\infty g(\varepsilon)\,[\text{step function}]\,d\varepsilon$ = Area under integrand

Eq.(C1) "reads":



$\boxed{\therefore \text{ Adjust value of } E_F \text{ so that Area} = N}$

Wrong guesses



The correct E_F gives the right N!

Fill in the Mathematics

$$N=\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \varepsilon^{1/2}\cdot[\text{Step function}]\,d\varepsilon$$

$$=\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^{E_F} \varepsilon^{1/2}\,d\varepsilon$$

unknown

← (C2) fixes E_F as a function of $\frac{N}{V}$

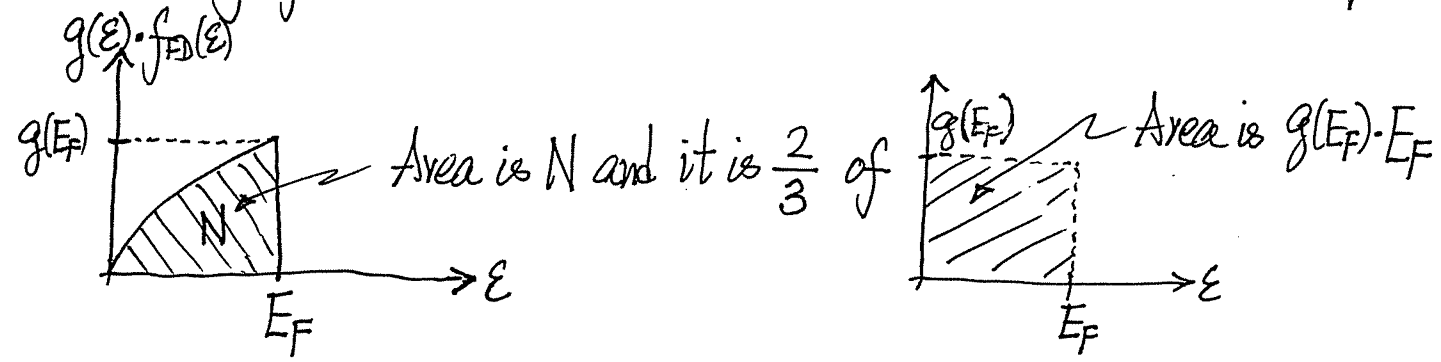
$$=\frac{2}{3}\cdot\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}E_F^{3/2} \qquad (*)$$

$$=\frac{2}{3}\cdot\left[\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}E_F^{1/2}\right]\cdot E_F$$

$$=\frac{2}{3}\,g(E_F)\cdot E_F \qquad (**)$$

$g(E_F)$ = density of states at the Fermi energy

Meaning of (**):



Back to (*): $N = \frac{2}{3} \cdot \frac{V}{2\pi}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{3/2}$

$$\Rightarrow E_F = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}$$

$$\Rightarrow \boxed{E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}} \quad (C3)$$

$\boxed{n \equiv \frac{N}{V} = \text{fermion (electron) number density}}$

$$\propto \left(\frac{N}{V}\right)^{2/3} \sim n^{2/3}$$

property of material

$[n_{Au} > n_{Na}] \Rightarrow$ different metals have different E_F

$\boxed{\text{order of } \sim 10^{22}\text{-}10^{23}\ cm^{-3} \text{ for metals}}$

$$E_F \sim n^{2/3}$$

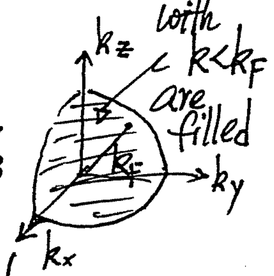
depends on $\frac{N}{V}$ $\Rightarrow$ Big piece of gold has the same E_F as a small piece of gold.

intensive

Form of Eq. (C3) leads us to define:

$$E_F = \frac{\hbar^2}{2m} k_F^2 \qquad k_F \equiv (3\pi^2 n)^{1/3} \sim n^{1/3}$$

$= \text{Fermi wave vector}$



and

$$E_F = kT_F \text{ OR } \boxed{T_F = \frac{E_F}{k} = \text{Fermi Temperature}} \quad (C4)$$

This is a zero-temperature property!

Eg. Metals

- $n = \frac{N}{V}$ = # conduction electrons per unit Volume of the metal

$$\sim 10^{22}\text{-}10^{23}/cm^3 \text{ OR } 10^{28}\text{-}10^{29}/m^3$$

$\therefore$ $E_F \sim$ a few eV from $E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$

$$T_F = \frac{E_F}{k} \sim 10^4 K$$

e.g. Cu $\quad n \sim 8.5 \times 10^{22}/cm^3$

$$E_F \sim 7\ eV$$

$$T_F \sim 8.2 \times \underbrace{10^4 K}$$

The physics of metals at room temperature OR ordinary "solid state physics temperatures" is low-temperature physics of an ideal Fermi gas!

Metals

$n \sim 10^{22} - 10^{23}/cm^3$

Different metals $\Rightarrow$ Different lattice constants / basis atoms

$\Rightarrow$ slightly different n

Fermi Velocity $E_F = \frac{1}{2} m v_F^2$ or $v_F = \frac{\hbar k_F}{m}$

Taken from Ashcroft and Mermin "Solid State Physics"

ELEMENT	E_F	T_F	k_F	v_F
Li	4.74 eV	5.51×10^4 K	1.12×10^8 cm^{-1}	1.29×10^8 cm/sec
Na	3.24	3.77	0.92	1.07
K	2.12	2.46	0.75	0.86
Rb	1.85	2.15	0.70	0.81
Cs	1.59	1.84	0.65	0.75
Cu	7.00	8.16	1.36	1.57
Ag	5.49	6.38	1.20	1.39
Au	5.53	6.42	1.21	1.40
Be	14.3	16.6	1.94	2.25
Mg	7.08	8.23	1.36	1.58
Ca	4.69	5.44	1.11	1.28
Sr	3.93	4.57	1.02	1.18
Ba	3.64	4.23	0.98	1.13
Nb	5.32	6.18	1.18	1.37
Fe	11.1	13.0	1.71	1.98
Mn	10.9	12.7	1.70	1.96
Zn	9.47	11.0	1.58	1.83
Cd	7.47	8.68	1.40	1.62
Hg	7.13	8.29	1.37	1.58
Al	11.7	13.6	1.75	2.03
Ga	10.4	12.1	1.66	1.92
In	8.63	10.0	1.51	1.74
Tl	8.15	9.46	1.46	1.69
Sn	10.2	11.8	1.64	1.90
Pb	9.47	11.0	1.58	1.83
Bi	9.90	11.5	1.61	1.87
Sb	10.9	12.7	1.70	1.96

Key Points:

- $T=0$ physics sets an energy scale of $E_F \sim$ few eV and a temperature scale $T_F \sim 10^4 - 10^5$ K.
- Room temperature metal physics has $kT \ll E_F$ or $T \ll T_F$

$\Rightarrow$ "low-temperature physics"!

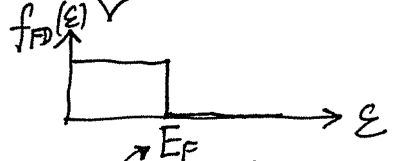
[Always compare temperature with a scale set by the problem!]

Total Energy E at T=0

$$E = \sum_{\text{s.p. states } i} \frac{\varepsilon_i}{e^{\beta(\varepsilon_i - \mu)} + 1} = \int_0^\infty g(\varepsilon) \frac{\varepsilon}{e^{\frac{\varepsilon-\mu}{kT}} + 1} d\varepsilon \quad (4a)$$

T=0

$$E = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \varepsilon^{1/2} \cdot \varepsilon \cdot \underbrace{[\text{Step function}]}_{} d\varepsilon$$



E_F now determined by Eq.(C1) to be Eq.(C3)

known

$$\Rightarrow \underset{\text{unknown}}{E} = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^{E_F} \varepsilon^{3/2} d\varepsilon \quad (C5)$$

$$= \frac{2}{5} \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{5/2} \propto V \cdot \left(\frac{N}{V}\right)^{5/3} \quad \text{(using Eq.(C3))}$$

$$= \frac{3}{5} \cdot \left[\frac{2}{3} \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{3/2}\right] \cdot E_F$$

$$\boxed{E = \frac{3}{5} \cdot N \cdot E_F} \quad (C6)$$

$$\Rightarrow \boxed{\frac{E}{N} = \underbrace{\frac{3}{5} E_F}_{}} \quad (C7)$$

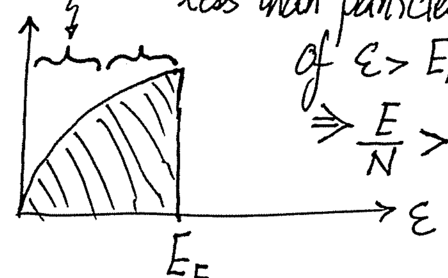
energy per particle at T=0

60% of E_F (high!)

[Pauli Principle]

particles in states of $\varepsilon < E_F/2$ is less than particles in states of $\varepsilon > E_F/2$

$\Rightarrow \frac{E}{N} > \frac{E_F}{2}$



(Reason: $g(\varepsilon)$ increases with ε)

Pressure p at $T=0$

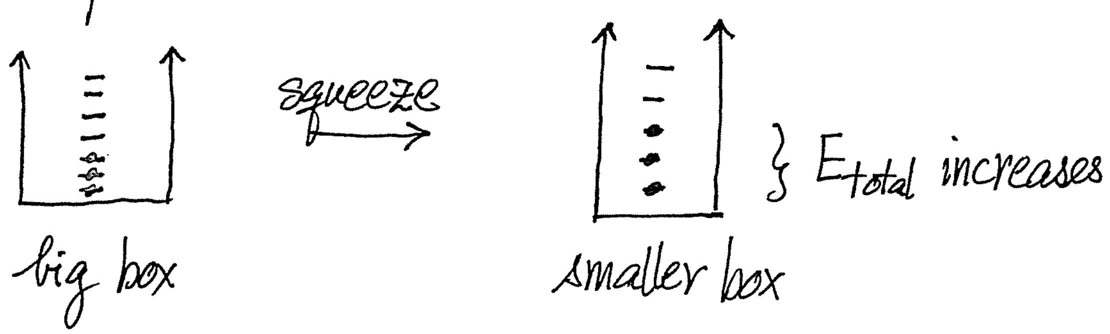
$$pV = \frac{2}{3}E = \frac{2}{3}\cdot\frac{3}{5}NE_F = \frac{2}{5}NE_F$$

$$\Rightarrow p = \frac{2}{5}\frac{N}{V}\cdot E_F = \frac{2}{5}\cdot\frac{N}{V}\cdot\frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3}$$

$$\Rightarrow \boxed{p \propto \left(\frac{N}{V}\right)^{5/3} \sim n^{5/3}} \qquad (C8)$$

- Recall this is a $T=0$ result
- p comes from Pauli Exclusion Principle[+]
  (due to piling up of fermions even at $T=0$)

1D picture:



$$\Delta V < 0 \text{ and } \Delta E > 0 \Rightarrow p = -\frac{\partial E}{\partial V} > 0$$

- This is called the <u>degenerate pressure</u>. This is the pressure that opposes the gravitational pull in astrophysics (evolution of stars).

[+] In contrast, for <u>classical</u> ideal gas $p = \frac{NkT}{V} \to 0$ as $T \to 0$ in thermodynamic limit ($\frac{N}{V}$ = finite).